PERGAMON

International Journal of Solids and Structures 37 (2000) 3535–3553

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

State space approach for stress decay in laminates

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Received 14 January 1998; received in revised form 13 February 1999

Abstract

Stress decay in laminates due to edge boundary effects are studied through a state space formulation. A self-equilibrium eigenstress field accounting for the multilayer construction of the laminate is derived using the state variables and the transfer matrix method. The eigenvalue determination requires only the solution of 6×6 determinants irrespective of the number of laminae. Through combinations of the eigenstress field and the interior stress field a complete solution valid in the boundary layer as well as in the interior region of the laminate can be obtained. For verification, the formulation is first applied to determining the eigenstress in a homogeneous anisotropic layer, and then the free edge stress decay in laminates under uniform extension is examined. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

Analysis of bending and stretching of plates is commonly based on a 2-D approximate theory in which the through-thickness variation of the displacement is assumed so that the coordinate in the thickness direction can be suppressed to derive the plate equations. Since the exact 3-D boundary conditions along the edge surfaces can hardly be satisfied within the framework of a 2-D plate theory, it is a common practice to require instead the edge boundary conditions be satisfied in their resultant forms. As such, the exact edge boundary conditions through the thickness are replaced by the edge conditions involving the midplane displacements, the stress resultants and stress moments. The solution thus obtained is known to be invalid within the edge boundary zone where the stress state is inherently 3-D. As far as the interior solution is concerned, nonetheless, it is generally expected, by virtue of Saint-Venant's principle, that the boundary layer effect will not be felt away from the local disturbance. The assertion of course needs to be examined.

Examination of the applicability of Saint-Venant's principle and boundary layer effects in elastic

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PII: S0020-7683(99)00050-5

strips and plates has received much attention (Toupin, 1965; Knowles, 1966; Choi and Horgan, 1977; 1978; Gregory and Wan, 1984; 1985; Crafter et al., 1993; Ting, 1996). It was found that the edge effects in anisotropic strips may be far reaching and the stress disturbance may not be local. The extent of the edge boundary layers depends on the geometry, the material property, the boundary conditions, and the applied load. For laminates the lamination scheme also plays an important role. To date numerous works on the subject have been published (see, e.g. Pagano,1974; Dong and Goetshel, 1982; Kassapoglou and Lagace, 1987; Becker, 1993; and the references therein), ranging from finite element solutions to analytic solutions based on various approximations and simplifications.

In this paper we develop a state space approach for analysis of stress decay in laminates due to edge effects. For simplicity, we limit considerations to rectangular laminates. The problem of stress singularities is not considered. The state space formulation has been used extensively in the area of optimal control (Derusso et al., 1965). Herein we employ the basic idea of state variables and transfer matrix to develop a simple and direct approach for the problem. In formulating a state space approach the first step is to express the field equations in the form of matrix differential equations in which the unknowns are the state variable vectors. For problems of laminated plates, it is advantageous to take the displacement and transverse stress components as the primary state variables because the continuity conditions at the interfaces and the boundary conditions on the top and bottom surfaces are directly associated with them. Under the assumption that the in-plane dimensions of the plate are large so that the edge boundary layers are not interactive, we first show that within the first-order approximation it suffices to consider the boundary layer in a laminated strip. Guided by previous studies on homogeneous strips (Toupin, 1965; Knowles, 1966; Choi and Horgan, 1977; Crafter et al., 1993), we then seek an eigenstress field that takes the form of exponential decay functions of the distance from the edge. The smallest decay factor is a measure of the rate of stress attenuation. With the assumed stress decay functions, the matrix differential equations for the state variables are reduced to a system of ordinary differential equations in which the coordinate in the thickness direction is the only independent variable. As a result, the state equations can be solved by means of matrix algebra (Frazer et al., 1960; Pease, 1965). The interfacial continuity conditions and the traction-free lateral boundary conditions are satisfied using a transfer matrix that transmits the state variables from the bottom to the top layer. The derivation produces a recursive relation that yields a self-equilibrium eigenstress field for the laminate. By combining the eigenstress field with the interior stress field that satisfies the prescribed boundary conditions on the top and bottom surfaces, we are able to satisfy the edge boundary conditions through the thickness for a specific problem and obtain a complete solution valid in the boundary layer as well as in the interior. The interior stress field can be determined in the usual way by the classical lamination theory (Jones, 1975; Whitney, 1987) or by more refined theories (Wang and Tarn, 1994; Tarn et al., 1996). Satisfaction of the edge boundary conditions through the thickness must resort to a numerical method in general.

The state space approach combined with the transfer matrix method provides a simple and systematic way for analysis of boundary layers in multilayered anisotropic laminates. Apart from being concise, the approach is effective in that the eigenvalue determination requires only the solution of 6×6 determinants irrespective of the number of laminae. By contrast, if one follows the usual layerwise approach, a very complicated $6n \times 6n$ (n is the number of laminae) determinant for the eigenvalue will result. The determinant involves the eigenvalue implicitly and often it is too large to solve, making the determination of eigensolutions for laminates virtually impossible.

In the next section we present the state space formulation of the problem and construct a self-equilibrium eigenstress field for multilayered anisotropic laminates. We shall verify the formulation by applying it to homogeneous layers in section 3. The eigensolutions derived herein fully agree with the known results. Finally, we shall use the approach to examine the edge stress decay in laminates subjected to uniform extension.

2. State space formulation

2.1. Basic equations

We consider a rectangular laminated plate composed of n anisotropic laminae having at each point one plane of elastic symmetry parallel to the midplane. The plate is subjected to (or free from) transverse loads on the top and bottom surfaces. Appropriate edge conditions through the thickness are prescribed on the edge boundary surfaces. The conditions are assumed to be uniform along the four edges.

Let us select a Cartesian coordinate system such that the plane coincides with the bottom surface. The stress-displacement relations for the laminae with respect to the laminate coordinates are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}_k = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix}_k \begin{Bmatrix} u_{1,\,1} \\ u_{2,\,2} \\ u_{3,\,3} \\ u_{2,\,3}+u_{3,\,2} \\ u_{1,\,3}+u_{3,\,1} \\ u_{1,\,2}+u_{2,\,1} \end{Bmatrix}_k, \tag{1}$$

where σ_{ij} are the stress components; u_i are the displacement components, the commas denote differentiation with respect to the suffix variables; c_{ij} are the 13 elastic constants of the material with one plane of material symmetry; the subscript k indicates the kth lamina ($k = 1,2\ \ldots,\ n$).

The stresses in each lamina must satisfy the usual equilibrium equations in addition to the continuity conditions between adjacent laminae at the interfaces:

$$(u_1)_k = (u_1)_{k+1}, \quad (u_2)_k = (u_2)_{k+1}, \quad (u_3)_k = (u_3)_{k+1},$$

$$(\sigma_{13})_k = (\sigma_{13})_{k+1}, \quad (\sigma_{23})_k = (\sigma_{23})_{k+1}, \quad (\sigma_{33})_k = (\sigma_{33})_{k+1}. \tag{2}$$

The boundary conditions on the top and bottom surfaces are

$$(\sigma_{13})_1 = (\sigma_{23})_1 = 0, \quad (\sigma_{33})_1 = q^-, \quad \text{at } x_3 = 0;$$

$$(\sigma_{13})_n = (\sigma_{23})_n = 0, \quad (\sigma_{33})_n = q^+, \quad \text{at } x_3 = h; \tag{3}$$

where $q^{\pm}=0$ if the top and bottom surfaces are free from the transverse loads.

The displacements and transverse stresses are taken to be the primary state variables in the state space formulation. Upon eliminating the in-plane stresses using (1) and the equilibrium equations, we can write the field equations for the kth lamina in the form

$$\frac{\partial}{\partial x_3}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{Bmatrix}_k = \begin{bmatrix} 0 & 0 & -\partial_1 & s_{55} & s_{45} & 0 \\ 0 & 0 & -\partial_2 & s_{45} & s_{44} & 0 \\ d_{31} & d_{32} & 0 & 0 & 0 & c_{33}^{-1} \\ d_{41} & d_{42} & 0 & 0 & 0 & d_{31} \\ d_{42} & d_{52} & 0 & 0 & 0 & d_{32} \\ 0 & 0 & 0 & -\partial_1 & -\partial_2 & 0 \end{bmatrix}_k \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{Bmatrix}_k, \tag{4}$$

where

$$d_{31} = -(c_{13}\partial_1 + c_{36}\partial_2)c_{33}^{-1}, \quad d_{32} = -(c_{36}\partial_1 + c_{23}\partial_2)c_{33}^{-1},$$

$$d_{41} = -(Q_{11}\partial_{11} + 2Q_{16}\partial_{12} + Q_{66}\partial_{22}),$$

$$d_{42} = -[Q_{16}\partial_{11} + (Q_{12} + Q_{66})\partial_{12} + Q_{26}\partial_{22}],$$

$$d_{52} = -(Q_{66}\partial_{11} + 2Q_{26}\partial_{12} + Q_{22}\partial_{22}), \quad Q_{ij} = c_{ij} - c_{i3}c_{j3}/c_{33}, \quad \begin{bmatrix} s_{55} & s_{45} \\ s_{45} & s_{44} \end{bmatrix} = \begin{bmatrix} c_{55} & c_{45} \\ c_{45} & c_{44} \end{bmatrix}^{-1},$$

∂_1, ∂_2, ∂_{12}, …, denote partial derivatives with respect to x_1, x_2, x_1 and x_2, …, respectively.

The in-plane stresses expressed in terms of the primary state variables are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}_k = \begin{bmatrix} Q_{11}\partial_1 + Q_{16}\partial_2 & Q_{16}\partial_1 + Q_{12}\partial_2 & c_{13}c_{33}^{-1} \\ Q_{12}\partial_1 + Q_{26}\partial_2 & Q_{26}\partial_1 + Q_{22}\partial_2 & c_{23}c_{33}^{-1} \\ Q_{16}\partial_1 + Q_{66}\partial_2 & Q_{66}\partial_1 + Q_{26}\partial_2 & c_{36}c_{33}^{-1} \end{bmatrix}_k \begin{Bmatrix} u_1 \\ u_2 \\ \sigma_{33} \end{Bmatrix}_k. \tag{5}$$

Without loss of generality, we shall focus our attention on the boundary layer at the edge $x_1 = 0$. To facilitate subsequent analysis, we introduce the stretched coordinates defined by

$$x = x_1/h, \quad y = x_2/l, \quad z = x_3/h, \tag{6}$$

where l denotes the length of the plate in the x_2 direction.

Upon substitution, Eqs. (4) and (5) become dimensionless as follows:

$$\frac{\partial}{\partial z}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{Bmatrix}_k = \begin{bmatrix} 0 & 0 & -\partial_x & s_{55} & s_{45} & 0 \\ 0 & 0 & -\epsilon\partial_y & s_{45} & s_{44} & 0 \\ d_{31} & d_{32} & 0 & 0 & 0 & c_{33}^{-1} \\ d_{41} & d_{42} & 0 & 0 & 0 & d_{31} \\ d_{42} & d_{42} & 0 & 0 & 0 & d_{32} \\ 0 & 0 & 0 & -\partial_x & -\epsilon\partial_y & 0 \end{bmatrix}_k \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{Bmatrix}_k, \tag{7}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}_k = \begin{bmatrix} Q_{11}\partial_x + \epsilon Q_{16}\partial_y & Q_{16}\partial_x + \epsilon Q_{12}\partial_y & c_{13}c_{33}^{-1} \\ Q_{12}\partial_x + \epsilon Q_{26}\partial_y & Q_{26}\partial_x + \epsilon Q_{22}\partial_y & c_{23}c_{33}^{-1} \\ Q_{16}\partial_x + \epsilon Q_{66}\partial_y & Q_{66}\partial_x + \epsilon Q_{26}\partial_y & c_{36}c_{33}^{-1} \end{bmatrix}_k \begin{Bmatrix} u_1 \\ u_2 \\ \sigma_{33} \end{Bmatrix}_k, \tag{8}$$

where

$$d_{31} = -(c_{13}\partial_x + \epsilon c_{36}\partial_y)c_{33}^{-1}, \quad d_{32} = -(c_{36}\partial_x + \epsilon c_{23}\partial_y)c_{33}^{-1},$$

$$d_{41} = -(Q_{11}\partial_{xx} + 2\epsilon Q_{16}\partial_{xy} + \epsilon^2 Q_{66}\partial_{yy}), \quad d_{42} = -(Q_{66}\partial_{xx} + 2\epsilon Q_{26}\partial_{xy} + \epsilon^2 Q_{22}\partial_{yy}),$$

$$d_{52} = -[Q_{16}\partial_{xx} + \epsilon(Q_{12} + Q_{66})\partial_{xy} + \epsilon^2 Q_{26}\partial_{yy}], \quad Q_{ij} = c_{ij} - c_{i3}c_{j3}/c_{33}.$$

The derivatives with respect to y in (7) and (8) are associated with $\epsilon(\epsilon = h/l < 1)$. As a first-order approximation, they can be neglected compared with the derivatives with respect to x. Thus Eqs. (7) and (8) become

$$\frac{\partial}{\partial z}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{Bmatrix}_k = \begin{bmatrix} 0 & 0 & -\partial_x & s_{55} & s_{45} & 0 \\ 0 & 0 & 0 & s_{45} & s_{44} & 0 \\ -c_{13}c_{33}^{-1}\partial_x & -c_{36}c_{33}^{-1}\partial_x & 0 & 0 & 0 & c_{33}^{-1} \\ -Q_{11}\partial_{xx} & -Q_{16}\partial_{xx} & 0 & 0 & 0 & -c_{13}c_{33}^{-1}\partial_x \\ -Q_{16}\partial_{xx} & -Q_{66}\partial_{xx} & 0 & 0 & 0 & -c_{36}c_{33}^{-1}\partial_x \\ 0 & 0 & 0 & -\partial_x & 0 & 0 \end{bmatrix}_k \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{Bmatrix}_k, \tag{9}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}_k = \begin{bmatrix} Q_{11}\partial_x & Q_{16}\partial_x & c_{13}c_{33}^{-1} \\ Q_{12}\partial_x & Q_{26}\partial_x & c_{23}c_{33}^{-1} \\ Q_{16}\partial_x & Q_{66}\partial_x & c_{36}c_{33}^{-1} \end{bmatrix}_k \begin{Bmatrix} u_1 \\ u_2 \\ \sigma_{33} \end{Bmatrix}_k. \tag{10}$$

Eqs. (9) and (10) are in fact the equations for a laminated elastic strip in which the field variables depend only on x and z. Consequently, at the first-order approximation, the problem in question is reduced to the analysis of a laminated elastic strip.

2.2. Eigensolution

We now seek a solution to (9) in the form

$$[u_1\ u_2\ u_3\ \sigma_{13}\ \sigma_{23}\ \sigma_{33}]_k^{\mathrm{T}} = \mathrm{e}^{-\lambda x}[U\ V\ W\ \tau_{13}\ \tau_{23}\ \tau_{33}]_k^{\mathrm{T}}, \tag{11}$$

where the components of the state vector $[U\ V\ W\ \tau_{13}\ \tau_{23}\ \tau_{33}]^{\mathrm{T}}$ are functions of z; λ is a factor indicating the decay rate from the edge boundary $x_1 = 0$. The values of λ can be determined through an eigenvalue problem.

Substituting (11) in (9), we write the resulting first-order ordinary differential equation system as

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathbf{X}_k = \lambda \mathbf{A}_k \mathbf{X}_k, \tag{12}$$

where

$$\mathbf{X}_k = [\lambda U\ \lambda V\ \lambda W\ \tau_{13}\ \tau_{23}\ \tau_{33}]_k^{\mathrm{T}},$$

$$\mathbf{A}_k = \begin{bmatrix} 0 & 0 & 1 & s_{55} & s_{45} & 0 \\ 0 & 0 & 0 & s_{45} & s_{44} & \\ c_{13}c_{33}^{-1} & c_{36}c_{33}^{-1} & 0 & 0 & 0 & c_{33}^{-1} \\ -Q_{11} & -Q_{16} & 0 & 0 & 0 & c_{13}c_{33}^{-1} \\ -Q_{16} & -Q_{66} & 0 & 0 & 0 & c_{36}c_{33}^{-1} \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}_k.$$

Eq. (10) becomes

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}_k = e^{-\lambda x} \begin{bmatrix} -Q_{11} & -Q_{16} & c_{13}c_{33}^{-1} \\ -Q_{12} & -Q_{26} & c_{23}c_{33}^{-1} \\ -Q_{16} & -Q_{66} & c_{36}c_{33}^{-1} \end{bmatrix}_k \begin{Bmatrix} \lambda U \\ \lambda V \\ \tau_{33} \end{Bmatrix}_k . \tag{13}$$

Note that (12) and (13) have been arranged in such a way that the coefficient matrices do not contain λ. The arrangement will make the eigenvalue determination much easier.

The solution of the matrix differential Eq. (12) (Frazer et al., 1960; Pease, 1965) takes the form

$$\mathbf{X}_k(z_k) = \mathbf{T}_k(z_k)\mathbf{X}_k(0), \tag{14}$$

where $0 \le z_k \le h_k$ ($h_k = t_k/h$, t_k denotes the thickness of the kth lamina); $\mathbf{T}_k(z_k)$ is the local transfer matrix defined by

$$\mathbf{T}_k(z_k) = e^{\lambda \mathbf{A}_k z_k}. \tag{15}$$

The relation between the local coordinate z_k and the global coordinate z is

$$z_k = z - \sum_{j=1}^{k-1} h_j. \tag{16}$$

The continuity conditions at the interfaces between the adjacent laminae require

$$\mathbf{X}_{k+1}(0) = \mathbf{X}_k(h_k). \tag{17}$$

With (14) and (17), we have the recursive relation

$$\mathbf{X}_{k+1}(z_k) = \mathbf{T}_{k+1}(z_k)\mathbf{T}_k(h_k)\mathbf{X}_k(0). \tag{18}$$

Carrying on the transformation from the bottom to the upper laminae using (18) and expressing the result in the global coordinate z, we obtain

$$\mathbf{X}(z) = \mathbf{T}(z)\mathbf{X}(0), \tag{19}$$

where the global transfer matrix is given by

$$\mathbf{T}(z) = \begin{cases} \mathbf{T}(z), & 0 \le z \le h_1; \\ \mathbf{T}_2(z - h_1)\mathbf{T}_1(h_1), & h_1 \le z \le h_1 + h_2; \\ \vdots & \vdots \\ \vdots & \vdots \\ \mathbf{T}_n(z - 1 + h_n)\mathbf{T}_{n-1}(h_{n-1}) \ldots \mathbf{T}_2(h_2)\mathbf{T}_1(h_1), & 1 - h_n \le z \le 1. \end{cases} \tag{20}$$

At the top surface $z = 1$, we have

$$\mathbf{X}(1) = \mathbf{T}(1)\mathbf{X}(0). \tag{21}$$

Eq. (21) can be written as

$$\begin{Bmatrix} \lambda U \\ \lambda V \\ \lambda W \\ \tau_{13} \\ \tau_{23} \\ \tau_{33} \end{Bmatrix}_{z=1} = \begin{bmatrix} \mathbf{T}_{\mathrm{uu}} & \mathbf{T}_{\mathrm{us}} \\ \mathbf{T}_{\mathrm{su}} & \mathbf{T}_{\mathrm{ss}} \end{bmatrix}_{z=1} \begin{Bmatrix} \lambda U \\ \lambda V \\ \lambda W \\ \tau_{13} \\ \tau_{23} \\ \tau_{33} \end{Bmatrix}_{z=0}, \tag{22}$$

where the transfer matrix has been partitioned such that

$$\begin{Bmatrix} \lambda U \\ \lambda V \\ \lambda W \end{Bmatrix}_{z=1} = [\mathbf{T}_{\mathrm{uu}}]_{z=1} \begin{Bmatrix} \lambda U \\ \lambda V \\ \lambda W \end{Bmatrix}_{z=0} + [\mathbf{T}_{\mathrm{us}}]_{z=1} \begin{Bmatrix} \tau_{13} \\ \tau_{23} \\ \tau_{33} \end{Bmatrix}_{z=0}, \tag{23}$$

$$\begin{Bmatrix} \tau_{13} \\ \tau_{23} \\ \tau_{33} \end{Bmatrix}_{z=1} = [\mathbf{T}_{\mathrm{su}}]_{z=1} \begin{Bmatrix} \lambda U \\ \lambda V \\ \lambda W \end{Bmatrix}_{z=0} + [\mathbf{T}_{\mathrm{ss}}]_{z=1} \begin{Bmatrix} \tau_{13} \\ \tau_{23} \\ \tau_{33} \end{Bmatrix}_{z=0}. \tag{24}$$

The eigensolution is constructed by considering traction-free boundary conditions on the top and bottom surfaces:

$$\begin{Bmatrix} \tau_{13} \\ \tau_{23} \\ \tau_{33} \end{Bmatrix}_{z=1} = \begin{Bmatrix} \tau_{13} \\ \tau_{23} \\ \tau_{33} \end{Bmatrix}_{z=0} = 0. \tag{25}$$

Substituting (25) in (24) gives

$$[\mathbf{T}_{\mathrm{su}}]_{z=1} \begin{Bmatrix} \lambda U \\ \lambda V \\ \lambda W \end{Bmatrix}_{z=0} = 0. \tag{26}$$

Non-trivial solutions of (26) exist if and only if the determinant of $[\mathbf{T}_{\mathrm{su}}]_{z=1}$ vanishes,

$$| \mathbf{T}_{\mathrm{su}} |_{z=1} = 0. \tag{27}$$

The solution of the characteristic Eq. (27) yields the decay factor λ. After determining λ, the eigenstress that satisfies the interfacial continuity conditions and the traction-free boundary conditions on the top and bottom surfaces is obtained from (19).

The transfer matrix given by (15) is a formal expression involving an exponential function of matrices. To use it in determining the eigensolution, we have to express (15) in an operational form. When the eigenvalues of $\mathbf{A}_k$ are distinct, it can be shown (Frazer et al., 1960; Pease, 1965), by making use of the Jordan canonical form, that the function of matrices can be expressed as

$$\mathbf{T}_k(z) = \mathrm{e}^{\lambda \mathbf{A}_k z} = \mathbf{M} \langle \mathrm{e}^{\lambda \mu z} \rangle \mathbf{M}^{-1}, \tag{28}$$

where $\langle \cdots \rangle$ denotes a diagonal matrix consisting of the six eigenvalues associated with the matrix $\mathbf{A}_k$; $\mathbf{M}$ is the matrix whose columns are the corresponding eigenvectors of $\mathbf{A}_k$. When repeated eigenvalues of occur, functions of matrices can be evaluated using a more general method (Frazer et al., 1960; Pease, 1965).

Determination of the eigenvalues of $\mathbf{A}_k$ requires the solution of a determinant. For the laminate, the eigenvalues are determined for each lamina in turn, and the transfer matrix defined by (20) is used for evaluating the decay factor. In the computation, we need to deal with only 6×6 determinants, one at a time irrespective of the number of laminae. The computation is further eased by the fact that the determinant depends only on the elastic moduli of the material, not on the decay factor λ. By contrast, if the usual layerwise approach is used, the governing equations for each lamina must be solved first, and then the interfacial continuity and lateral boundary conditions are imposed on the solution for the unknown coefficients. The approach will inevitably lead to a $6n \times 6n$ determinant for the eigenvalues. Not only is the determinant often too large to solve but it involves the decay factor implicitly, making it very difficult to treat the problem for multilayered laminates this way.

3. Eigenstress in a homogeneous layer

Before applying the state space formulation to multilayered laminates, we first apply it to problems of homogeneous strips. The eigensolution derived herein can be checked against the known results for isotropic and anisotropic elastic strips (Timoshenko and Goodier, 1970; Ting, 1996). In what follows we shall drop the subscript k for clarity.

For a homogeneous layer the eigenvalues and eigenvectors of $\mathbf{A}$ are derived from

$$\mathbf{AX} = \mu\mathbf{x}. \tag{29}$$

Non-trivial solution of (29) exists if

$$\begin{vmatrix} -\mu & 0 & 1 & s_{55} & s_{45} & 0 \\ 0 & -\mu & 0 & s_{45} & s_{44} & 0 \\ c_{13}c_{33}^{-1} & c_{36}c_{33}^{-1} & -\mu & 0 & 0 & c_{33}^{-1} \\ -Q_{11} & -Q_{16} & 0 & -\mu & 0 & c_{13}c_{33}^{-1} \\ -Q_{16} & -Q_{66} & 0 & 0 & -\mu & c_{36}c_{33}^{-1} \\ 0 & 0 & 0 & 1 & 0 & -\mu \end{vmatrix} = 0. \tag{30}$$

This gives the characteristic equation for the eigenvalues μ of monoclinic materials:

$$\begin{aligned} &\mu^6 + (s_{44}Q_{66} + s_{55}Q_{11} - 2c_{13}c_{33}^{-1})\mu^4 + [(s_{44}s_{55} - s_{45}^2)(Q_{11}Q_{66} - Q_{16}^2) + 2s_{44}(c_{36}Q_{16} - c_{13}Q_{66})c_{33}^{-1} \\ &\quad + 2s_{45}(c_{36}Q_{11} - c_{13}Q_{16})c_{33}^{-1} + c_{13}^2c_{33}^{-2} + Q_{11}c_{33}^{-1}]\mu^2 + s_{44}[c_{36}^2c_{33}^{-2}Q_{11} + c_{13}^2c_{33}^{-2}Q_{66} - 2c_{13}c_{16}c_{36}c_{33}^{-2} \\ &\quad + (Q_{11}Q_{66} - Q_{16}^2)c_{33}^{-1}] = 0. \end{aligned} \tag{31}$$

For orthotropic materials ($c_{16} = c_{26} = c_{36} = s_{45} = Q_{16} = 0$), Eq. (31) becomes

$$(\mu^2 + c_{66}c_{44}^{-1})\{\mu^4 + [c_{11}c_{55}^{-1} - (2 + c_{13}c_{55}^{-1})c_{13}c_{33}^{-1}]\mu^2 + c_{11}c_{33}^{-1}\} = 0. \tag{32}$$

For isotropic materials, the equation is further reduced to

$$(\mu^2 + 1)^3 = 0. \tag{33}$$

After solving the algebraic equations for the eigenvalues, we can easily obtain the corresponding eigenvectors from (29). According to (28), determination of the transfer matrix requires an inverse of the matrix of the eigenvectors. The inverse, of course, can be determined numerically for a given material, leading to numerical results for the stress decay factor. Nonetheless, we shall derive the analytic form of the stress decay factor using certain orthogonality properties for the eigenvalue problem.

The matrix **A** can be written as

$$\mathbf{A} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^{\mathrm{T}} \end{bmatrix}, \tag{34}$$

in which

$$\mathbf{N}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ c_{13}c_{33}^{-1} & c_{36}c_{33}^{-1} & 0 \end{bmatrix},$$

$$\mathbf{N}_2 = \begin{bmatrix} s_{55} & s_{45} & 0 \\ s_{45} & s_{44} & 0 \\ 0 & 0 & c_{33}^{-1} \end{bmatrix},$$

$$\mathbf{N}_3 = \begin{bmatrix} -Q_{11} & -Q_{16} & 0 \\ -Q_{16} & -Q_{66} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$\mathbf{N}_2$ and $\mathbf{N}_3$ are symmetric matrix.

The roots of (31) are three pairs of complex conjugate. Suppose that all the roots are distinct, we can represent the matrix **M** as

$$\mathbf{M} = \begin{bmatrix} \mathbf{B} & \bar{\mathbf{B}} \\ \mathbf{C} & \bar{\mathbf{C}} \end{bmatrix}, \tag{35}$$

where $\mathbf{B} = [\mathbf{b}_1\ \ \mathbf{b}_2\ \ \mathbf{b}_3]$, $\mathbf{C} = [\mathbf{c}_1\ \ \mathbf{c}_2\ \ \mathbf{c}_3]$, $\bar{\mathbf{B}}$ and $\bar{\mathbf{C}}$ are their complex conjugates. Henceforth, an over bar denotes the complex conjugate.

The components of **B** and **C** are the right and left eigenvectors associated with the eigenvalues $\mu_i (i = 1, 2, \ldots, 6)$. They are determined from

$$\mathbf{A}\mathbf{X}_i = \mu_i \mathbf{X}_i \quad \text{and} \quad \mathbf{Y}_i^{\mathrm{T}}\mathbf{A} = \mu_i \mathbf{Y}_i^{\mathrm{T}} \tag{36}$$

such that

$$\mathbf{X}_i = \begin{Bmatrix} \mathbf{b}_i \\ \mathbf{c}_i \end{Bmatrix}, \quad \mathbf{Y}_i = \begin{Bmatrix} \mathbf{c}_i \\ \mathbf{b}_i \end{Bmatrix}.$$

It can be shown by using the orthogonality property of the right and left eigenvectors (Ting, 1996) that the inverse of **M** is given by

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{C}^{\mathrm{T}} & \mathbf{B}^{\mathrm{T}} \\ \bar{\mathbf{C}}^{\mathrm{T}} & \bar{\mathbf{B}}^{\mathrm{T}} \end{bmatrix}. \tag{37}$$

Upon substituting (35) and (37) in (28) for the transfer matrix, we obtain

$$\mathbf{T}_{\mathrm{su}} = \mathbf{C}\langle \mathrm{e}^{\lambda\mu z} \rangle \mathbf{C}^{\mathrm{T}} + \bar{\mathbf{C}}\langle \mathrm{e}^{\lambda\bar{\mu} z} \rangle \bar{\mathbf{C}}^{\mathrm{T}}. \tag{38}$$

Eq. (38) can be manipulated to a simpler form using the closure relation for the components of **M** and $\mathbf{M}^{-1}$:

$$\mathbf{C}\mathbf{C}^{\mathrm{T}} + \bar{\mathbf{C}}\bar{\mathbf{C}}^{\mathrm{T}} = 0, \tag{39}$$

then

$$\mathbf{C}^{\mathrm{T}}(\bar{\mathbf{C}}^{\mathrm{T}})^{-1} = -\mathbf{C}^{-1}\bar{\mathbf{C}}. \tag{40}$$

Thus we have

$$\mathbf{T}_{\mathrm{su}} = \mathbf{C}[\langle \mathrm{e}^{\lambda\mu z}\rangle \mathbf{C}^{\mathrm{T}}(\bar{\mathbf{C}}^{\mathrm{T}})^{-1} + \mathbf{C}^{-1}\bar{\mathbf{C}}\langle \mathrm{e}^{\lambda\bar{\mu}z}\rangle]\bar{\mathbf{C}}^{\mathrm{T}} = -\mathbf{C}[\langle \mathrm{e}^{\lambda\mu z}\rangle \mathbf{C}^{-1}\bar{\mathbf{C}} - \mathbf{C}^{-1}\bar{\mathbf{C}}\langle \mathrm{e}^{\lambda\bar{\mu}z}\rangle]\bar{\mathbf{C}}^{\mathrm{T}}. \tag{41}$$

The condition (27) requires that

$$|\ \langle \mathrm{e}^{\lambda\mu}\rangle \mathbf{C}^{-1}\bar{\mathbf{C}} - \mathbf{C}^{-1}\bar{\mathbf{C}}\langle \mathrm{e}^{\lambda\bar{\mu}}\rangle\ | = 0. \tag{42}$$

From (42) the stress decay factor λ can be determined once the eigenvalues and eigenvectors of **A** are obtained.

Consider a single layer of orthotropic materials for example. The characteristics equations of **A** is given by (32) which can easily be solved to yield the six eigenvalues. The eigenvectors associated with the eigenvalues $\mu_{1,2} = \pm i(c_{66}/c_{44})^{1/2}$ are the antiplane mode of which the only non-zero components are

$$\begin{Bmatrix} b \\ c \end{Bmatrix} = \begin{Bmatrix} (1 \mp i)(c_{66}c_{44})^{-1/4}/2 \\ (1 \pm i)(c_{66}c_{44})^{1/4}/2 \end{Bmatrix}. \tag{43}$$

Thus

$$\mathbf{M} = \begin{bmatrix} (1-i)(c_{66}c_{44})^{-1/4}/2 & (1+i)(c_{66}c_{44})^{-1/4}/2 \\ (1+i)(c_{66}c_{44})^{1/4}/2 & (1-i)(c_{66}c_{44})^{1/4}/2 \end{bmatrix}, \tag{44}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} (1+i)(c_{66}c_{44})^{1/4}/2 & (1-i)(c_{66}c_{44})^{-1/4}/2 \\ (1-i)(c_{66}c_{44})^{1/4}/2 & (1+i)(c_{66}c_{44})^{-1/4}/2 \end{bmatrix}. \tag{45}$$

Substituting (44) and (45) in (28), we obtain the transfer matrix

$$\mathbf{T}(z) = \begin{bmatrix} \cos(\alpha\lambda z) & \sin(\alpha\lambda z)/\alpha \\ -\alpha\ \sin(\alpha\lambda z) & \cos(\alpha\lambda z) \end{bmatrix}, \tag{46}$$

where $\alpha = (c_{66}/c_{44})^{1/2}$.

The condition (27) demands

$$|\ \alpha\ \ \sin(\alpha\lambda)\ | = 0 \tag{47}$$

from which we obtain the stress decay factor for the antiplane mode:

$$\lambda = n\pi/\alpha = n\pi(c_{44}/c_{66})^{1/2}, \quad (n = 1, 2, \ldots) \tag{48}$$

and the corresponding eigenstate

$$[V \quad \tau_{23}] = [\,\cos(\alpha\lambda z)/\lambda \quad -\alpha\ \sin(\alpha\lambda z)]. \tag{49}$$

Next, we consider the other eigenvalues determined from (32). Let us denote the four roots of

$$\mu^4 + [c_{11}c_{55}^{-1} - (2 + c_{13}c_{55}^{-1})c_{13}c_{33}^{-1}]\mu^2 + c_{11}c_{33}^{-1} = 0 \tag{50}$$

by μ_1, μ_2, $\bar{\mu}_1$, $\bar{\mu}_2$.

The eigenvectors associated with these eigenvalues are the in-plane mode, and

$$\mathbf{M} = \begin{bmatrix} \mathbf{B} & \bar{\mathbf{B}} \\ \mathbf{C} & \bar{\mathbf{C}} \end{bmatrix} = \begin{bmatrix} r_1/k_1 & r_2/k_2 & \bar{r}_1/\bar{k}_1 & \bar{r}_2/\bar{k}_2 \\ s_1/k_1 & s_2/k_2 & \bar{s}_1/\bar{k}_1 & \bar{s}_2/\bar{k}_2 \\ \mu_1/k_1 & \mu_2/k_2 & \bar{\mu}_1/\bar{k}_1 & \bar{\mu}_2/\bar{k}_2 \\ 1/k_1 & 1/k_2 & 1/\bar{k}_1 & 1/\bar{k}_2 \end{bmatrix}, \tag{51}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{C}^{\mathrm{T}} & \mathbf{B}^{\mathrm{T}} \\ \bar{\mathbf{C}}^{\mathrm{T}} & \bar{\mathbf{B}}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \mu_1/k_1 & 1/k_1 & r_1/k_1 & s_1/k_1 \\ \mu_2/k_2 & 1/k_2 & r_2/k_2 & s_2/k_2 \\ \bar{\mu}_1/\bar{k}_1 & 1/\bar{k}_1 & \bar{r}_1/\bar{k}_1 & \bar{s}_1/\bar{k}_1 \\ \bar{\mu}_2/\bar{k}_2 & 1/\bar{k}_2 & \bar{r}_2/\bar{k}_2 & \bar{s}_2/\bar{k}_2 \end{bmatrix}, \tag{52}$$

where

$$r_i = (c_{13} + c_{55})\mu_i^2/[c_{55}(c_{13}\mu_i^2 - c_{11})],$$

$$s_i = (c_{55}\mu_i^2 + c_{11})\mu_i/[c_{55}(c_{13}\mu_i^2 - c_{11})],$$

$$k_i = \{2[c_{13} + 2c_{55})\mu_i^2 + c_{11}]\mu_i/[c_{55}(c_{13}\mu_i^2 - c_{11})]\}^{1/2} = [2(r_i\mu_i + s_i)]^{1/2}.$$

The condition (27) results in

$$\begin{vmatrix} (\bar{\mu}_1 - \mu_2)(\mathrm{e}^{\lambda\mu_1} - \mathrm{e}^{\lambda\bar{\mu}_1}) & (\bar{\mu}_2 - \mu_2)(\mathrm{e}^{\lambda\mu_1} - \mathrm{e}^{\lambda\bar{\mu}_2}) \\ (\mu_1 - \bar{\mu}_1)(\mathrm{e}^{\lambda\mu_2} - \mathrm{e}^{\lambda\bar{\mu}_1}) & (\mu_1 - \bar{\mu}_2)(\mathrm{e}^{\lambda\mu_2} - \mathrm{e}^{\lambda\bar{\mu}_2}) \end{vmatrix} = 0. \tag{53}$$

The determinant yields

$$(\alpha^2 + \beta^2)\sin^2\hat{\lambda} = (1 + \alpha^2)\sin(\hat{\lambda}\beta) + (1 - \beta^2)\sinh^2(\hat{\lambda}\alpha), \tag{54}$$

where

$$\hat{\lambda} = \lambda(q_1 + q_2)/2,$$

$$\alpha = (p_1 - p_2)/(q_1 + q_2),$$

$$\beta = (q_1 - q_2)/(q_1 + q_2),$$

p_i and q_i are the real and imaginary parts of the complex root μ_i.

Apart from notational differences, Eq. (54) is precisely the characteristic equation for the decay factor of the in-plane mode given in Ting (1996). The existing solution (Wang, et al., 1993; Ting, 1996) was derived based on the Stroh formalism for an anisotropic elastic strip.

The above derivations assume that the matrix **A** is simple or semisimple (Pease, 1965; Ting, 1996) such that there exist six independent eigenvectors. For isotropic materials, the matrix **A** is non-semisimple. The roots of (33) are $\mu = \pm i$ of multiplicity three. By setting $c_{44} = c_{66}$ in (48), we immediately obtain the stress decay factor for the antiplane mode

$$\lambda = n\pi, \quad (n = 1, 2 \ldots). \tag{55}$$

The stress decay factor of the in-plane mode can be obtained by letting $\mu_1 = i$ and $\mu_2 = (1+\epsilon)i$ $(\epsilon \to 0)$ in (54) and taking the limit. As a result of the limiting process, we obtain the following equation for λ:

$$\sin \lambda \pm \lambda = 0. \tag{56}$$

The characteristic Eq. (56) is the same as that obtained according to the stress function formulation for plane problems of elasticity (Timoshenko and Goodier, 1970). The analysis shows that the present formulation when applied to special cases yields results in full agreement with the known results.

4. Laminates under uniform extension

We now consider the edge effect of cross-ply laminates subjected to uniform tension in the x_2 direction. The laminate is composed of orthotropic laminae with midplane symmetry. The stress disturbance near the free edge $x_1 = 0$ is examined.

According to the classical lamination theory (CLT) (Jones, 1975; Whitney, 1987), uniform extension of a midplane symmetric laminate causes in-plane stresses:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}_k = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix}_k \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ N \\ 0 \end{Bmatrix}, \tag{57}$$

$$[\sigma_{13}\ \sigma_{23}\ \sigma_{33}]_k = 0, \tag{58}$$

where

$$A_{ij} = \sum_{k=1}^{n} (Q_{ij})_k t_k.$$

The free-edge boundary conditions require $\sigma_{11} = \sigma_{12} = \sigma_{13} = 0$ at $x_1 = 0$, whereas the stress distribution based on CLT was obtained using the relaxed conditions such that the stress resultants across the thickness vanish at the free edge. Thus the free-edge boundary conditions are satisfied only in an average sense, and the stress state given by (57) and (58) is invalid in the boundary layer zone.

Let us denote the eigenstate derived by Φ_i^σ and Φ_i^{u}, and the stress and displacement in the interior region by σ^{I} and $\mathbf{u}^{\mathrm{I}}$. The linear combination of the interior stress state and the eigenstress state can be expressed as

$$\sigma = \sigma^{\mathrm{I}} + \sum_i c_i \Phi_i^\sigma, \quad \mathbf{u} = \mathbf{u}^{\mathrm{I}} + \sum_i c_i \Phi_i^{\mathrm{u}}. \tag{59}$$

Obviously, the combined stress and displacement fields satisfy the equilibrium, the compatibility, the interfacial continuity conditions, and the boundary conditions on the top and bottom surfaces of the laminate. It remains to determine the unknown coefficients c_i in (59) such that the free-edge boundary conditions through the thickness are satisfied.

In determining the coefficients c_i, it would be possible to impose the free-edge conditions directly on (59) and use a numerical scheme such as the method of the least square error to find the values of c_i. We propose instead to determine c_i through the principle of virtual work. With a statically admissible elastic field in the laminate where only the edge boundary conditions are not satisfied, the virtual work equation is

$$\int_0^h \int_\Gamma (\sigma_{ij} n_j - p_i)\delta u_i \, d\Gamma \, dx_3 = 0, \tag{60}$$

where n_i is the outward normal to the edge contour Γ, p_i is the traction on the edge boundary. For traction-free edge boundary conditions, $p_i = 0$.

Substituting (59) in (60), we have

$$\sum_{i=1}^{n} c_i \int_0^1 (\Phi_i^\sigma \mathbf{n})^{\mathrm{T}} \Phi_j^{\mathrm{u}} \, dz + \int_0^1 (\sigma^{\mathrm{I}} \mathbf{n})^{\mathrm{T}} \Phi_j^{\mathrm{u}} \, dz = 0, \quad (j = 1, 2, \ldots, n) \tag{61}$$

where **n** denotes the outward normal to the edge contour.

Eq. (61) is a system of linear algebraic equations in which c_i are the unknowns. The equations can be easily solved using a numerical method. Upon determining the coefficients c_i, we obtain a complete solution valid in the edge boundary zone as well as in the interior region of the laminate.

To check the validity of the eigensolution, we first applied the present approach to four-ply cross-ply laminates, for which numerical results on the characteristic decay length were obtained by Dong and Goetschel (1982) using a semianalytical method with finite element interpolations over the thickness. The elastic constants for the lamina are $c_{11} = 21.289 \times 10^6$ psi, $c_{22} = c_{33} = 2.319 \times 10^6$ psi, $c_{44} = c_{55} = 0.85 \times 10^6$ psi, $c_{23} = 5.005 \times 10^6$ psi, $c_{12} = c_{13} = 0.592 \times 10^6$ psi. In their finite element model 40 equal thickness elements with 81 nodal surfaces and 162 degrees of freedom were used. Numerical results on the eigenvalues and eigenmodes for two cross-ply laminates with stacking sequences of [0/90/0/90] and [0/90/90/0] lay-up were reported. The comparisons of the lowest 20 eigenvalues λ_n for the two laminates, as determined via the state space approach with the numerical results reported in Dong and Goetschel (1982) were given in Table 1 and 2. The results are in good agreement. The eigenmodes for the [0/90/90/0] laminate may be identified as symmetric and antisymmetric about the middle plane. The lowest eigenvalue gives a measure of the stress decay rate. The real part of the lowest eigenvalue is 2.672 for the [0/90/90/0] laminate and 2.3713 for the [0/90/0/90] laminate, as compared with the known value of

Table 1
Decay rate λ_n for a [0/90/90/0] laminate

	Symmetric mode				Anti-symmetric mode			
	Present		Dong and Goetschel (1982)		Present		Dong and Goetschel (1982)	
Mode	Re[λ_n]	Im[λ_n]	Re[λ_n]	Im[λ_n]	Re[λ_n]	Im[λ_n]	Re[λ_n]	Im[λ_n]
1	2.6720	0.0000	2.6720	0.0000	2.6734	0.0000	2.6734	0.0000
2	4.4345	0.0000	4.4350	0.0000	5.0527	0.0000	5.0532	0.0000
3	4.6616	0.8087	4.6617	0.8085	8.0073	0.0000	8.0132	0.0000
4	4.6616	−0.8087	4.6617	−0.8085	9.1794	1.5306	9.1784	1.5292
5	7.8942	0.0000	7.8981	0.0000	9.1794	−1.5306	9.1784	−1.5292
6	10.2012	0.0000	10.2160	0.0000	10.0678	0.0000	10.0859	0.0000
7	13.5467	0.0000	13.6429	0.0000	13.1249	0.0000	13.1716	0.0000
8	13.9886	1.3853	13.9674	1.3830	15.3849	0.0000	15.4914	0.0000
9	13.9886	−1.3853	13.9674	−1.3830	17.9128	1.1077	17.9880	1.1409
10	15.3395	0.0000	15.4637	0.0000	17.9128	−1.1077	17.9880	−1.1409

Table 2
Decay rate λ_n for a [0/90/0/90] laminate

Mode	Present		Dong and Goetschel (1982)	
	Re[λ_n]	Im[λ_n]	Re[λ_n]	Im[λ_n]
1	2.3713	0.0000	2.3713	0.0000
2	2.6933	0.0000	2.6933	0.0000
3	4.1402	0.0000	4.1404	0.0000
4	5.0165	0.0000	5.0169	0.0000
5	5.0597	1.0254	5.0598	1.0254
6	5.0597	−1.0254	5.0598	−1.0254
7	7.1784	0.0000	7.1810	0.0000
8	7.8031	0.0000	7.8070	0.0000
9	8.8630	0.0000	8.8684	0.0000
10	10.2268	0.0000	10.2423	0.0000
11	10.9951	1.7760	10.9986	1.7795
12	10.9951	−1.7760	10.9986	−1.7795
13	12.6906	0.0000	12.7373	0.0000
14	12.9096	0.0000	12.9580	0.0000
15	13.0324	2.5474	13.0261	2.5468
16	13.0324	−2.5474	13.0261	−2.5468
17	14.9695	0.0000	15.0750	0.0000
18	15.4383	0.0000	15.5499	0.0000
19	17.9235	0.8710	17.9740	0.9408
20	17.9235	−0.8710	17.9740	−0.9408

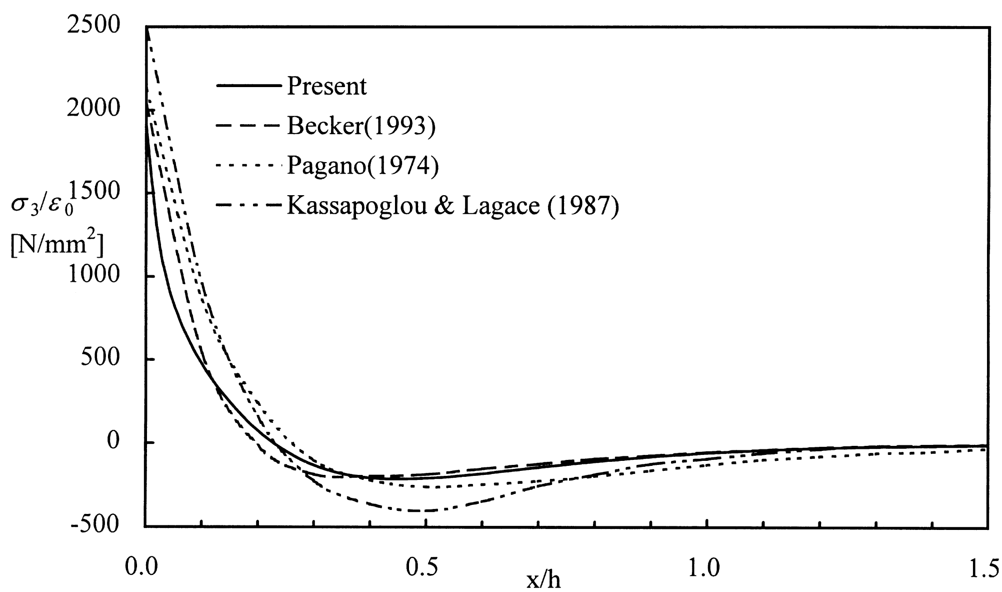

Fig. 1. Comparisons of the interlaminar normal stress for a $[0/90]_s$ laminate under extension.

Table 3
Decay rate λ_n for a $[0/90]_s$ laminate

Mode	Symmetric mode		Anti-symmetric mode	
	Re[λ_n]	Im[λ_n]	Re[λ_n]	Im[λ_n]
1	2.4047	0.0000	2.8255	0.0000
2	4.8382	1.3184	4.5541	0.0000
3	4.8382	−1.3184	6.8598	0.0000
4	5.0570	0.0000	9.4984	0.0000
5	7.9168	0.0000	11.1400	3.0460
6	10.5279	0.0000	11.1400	−3.0460
7	13.2253	0.0000	12.3833	0.0000
8	12.9979	4.0733	15.1680	0.0000
9	12.9979	−4.0733	17.3198	1.0840
10	15.9337	0.0000	17.3198	−1.0840
11	18.5857	0.0000	19.5944	0.0000
12	21.2594	0.0000	22.3651	0.0000
13	24.0235	0.0000	25.0737	0.0000
14	24.1721	5.6237	27.7725	0.0000
15	24.1721	−5.6237	30.4495	0.0000
16	27.1007	0.0000	33.1446	0.0000
17	28.5040	0.9782	35.9655	0.0000
18	28.5040	−0.9782	38.6005	1.1595
19	31.5260	0.0000	38.6005	−1.1595
20	34.2659	0.0000	40.5771	0.0000
21	36.9633	0.0000	43.4662	0.0000
22	37.3105	7.4102	46.1719	0.0000
23	37.3105	−7.4102	48.8610	0.0000
24	39.6563	0.0000	51.5551	0.0000
25	42.3446	0.0000	54.2617	0.0000
26	45.0738	0.0000	57.1002	0.0000
27	48.6610	0.0000	59.1921	1.0403
28	48.9609	0.6276	59.1921	−1.0403
29	48.9609	−0.6276	61.7777	0.0000
30	52.6382	0.0000	64.5859	0.0000
31	55.3861	0.0000	67.2943	0.0000
32	58.0752	0.0000	69.9763	0.0000
33	60.7622	0.0000	72.6676	0.0000
34	63.4663	0.0000	74.9275	7.7736
35	66.2189	0.0000	74.9275	−7.7736
36	69.2062	0.8647	75.3988	0.0000
37	69.2062	−0.8647	78.3376	0.0000
38	70.8249	0.0000	80.0243	0.9013
39	73.7720	0.0000	80.0243	−0.9013
40	75.3863	7.8345	82.9475	0.0000

4.212 for isotropic materials, indicating that the stress decay is slower in the laminate than in an isotropic material.

Next, we computed the results using another set of data for a $[0/90]_s$ laminate, for which the results based on various approximate solutions (Pagano,1974; Kassapoglou and Lagace, 1987; Becker, 1993) are available for comparisons. The elastic constants for the lamina are $c_{11} = c_{33} = 15{,}300$ N/mm^2, $c_{22} = 140{,}000$ N/mm^2, $c_{44} = c_{55} = 5900$ N/mm^2, $c_{12} = c_{23} = 3900$ N/mm^2, $c_{13} = 3300$ N/mm^2. The numerical results for the lowest forty eigenvalues are listed in Table 3. Again, the eigenvalues may be identified as

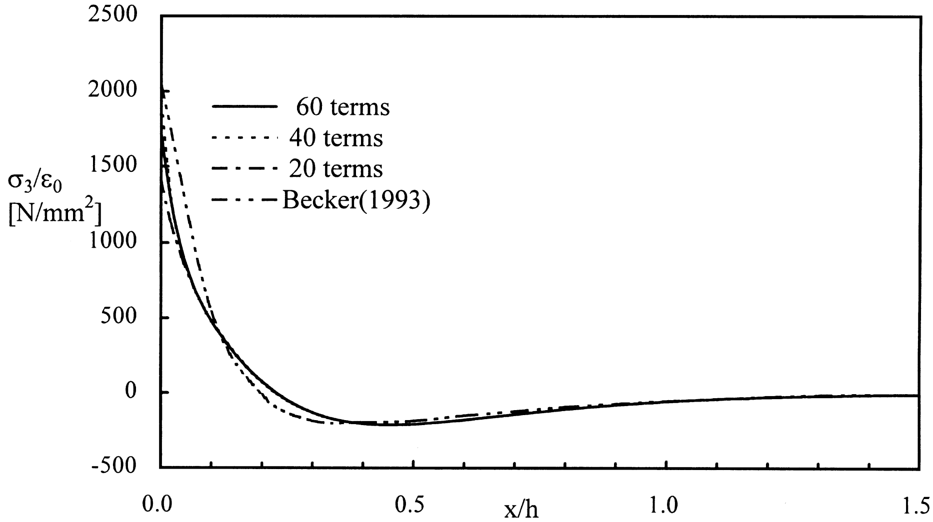

Fig. 2. Interlaminar normal stress σ_3/ϵ_0 at the 0/90-interface.

symmetric and antisymmetric modes about the middle plane. The real part of the lowest eigenvalue is 2.405 compared with the known value of 4.212 for isotropic materials. Fig. 1 shows the comparison of the interlaminar normal stress at the 0/90-interface with the published results. The results obtained by Becker (1993) using an assumed displacement model with a particular warp deformation mode are relatively close to the present results. In Figs. 2 and 3, we further compare the distributions of the interlaminar normal stress σ_3/ϵ_0 and shear stress σ_{13}/ϵ_0 (ϵ_0 being the uniform strain in the x_2 direction) at the 0/90-interface away from the free edge. The results by taking 20, 40, and 60 terms in the eigenfunction expansions are also presented to show the convergence. It is found that satisfactory

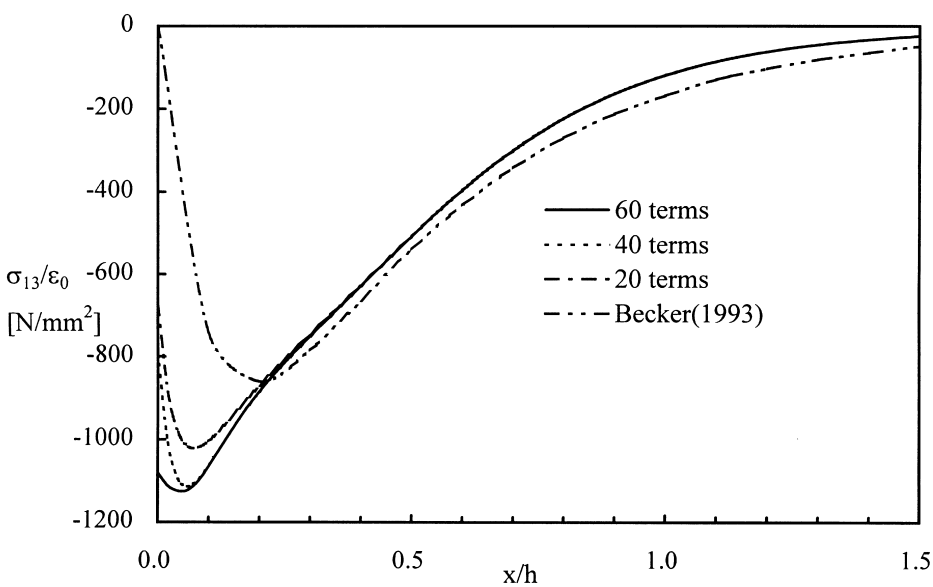

Fig. 3. Interlaminar shear stress σ_{13}/ϵ_0 at the 0/90-interface.

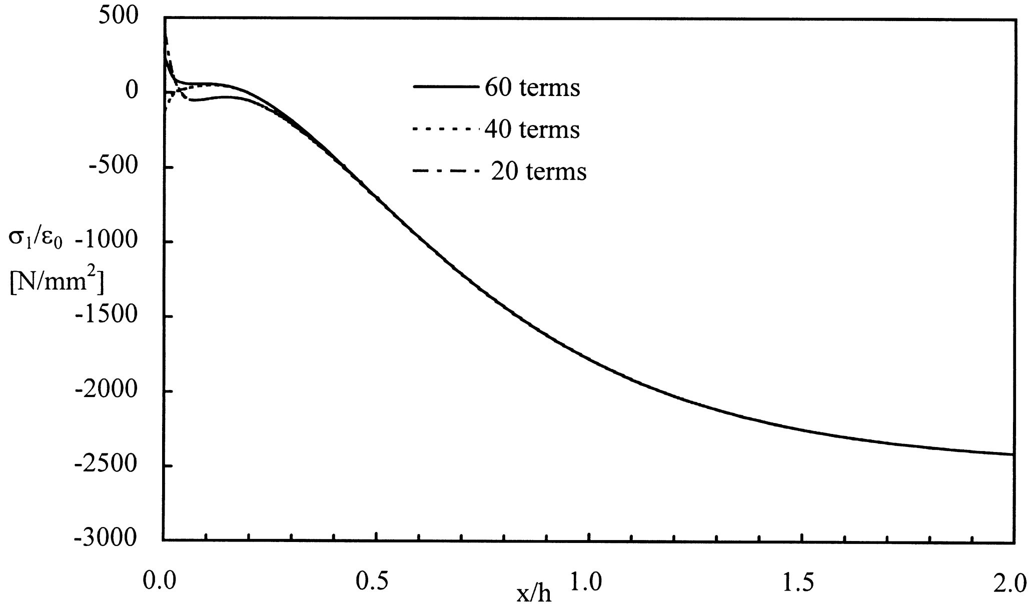

Fig. 4. In-plane stress σ_1/ϵ_0 at the middle surface of the laminate.

convergence is reached except in a very narrow region ($x_1 < 0.1\ h$) near the free edge. While the present results and Becker's results are in reasonable agreement in the region $x_1 > 0.25\ h$, marked discrepancies exist within $x_1 < 0.25\ h$. According to CLT the normal stress σ_3 and shear stress σ_{13} vanish everywhere in symmetric laminates under extension, but Figs. 2 and 3 show that these stresses approach to zero only for $x_1 > 1.5\ h$, suggesting that stress disturbance occurs near the free edge. The

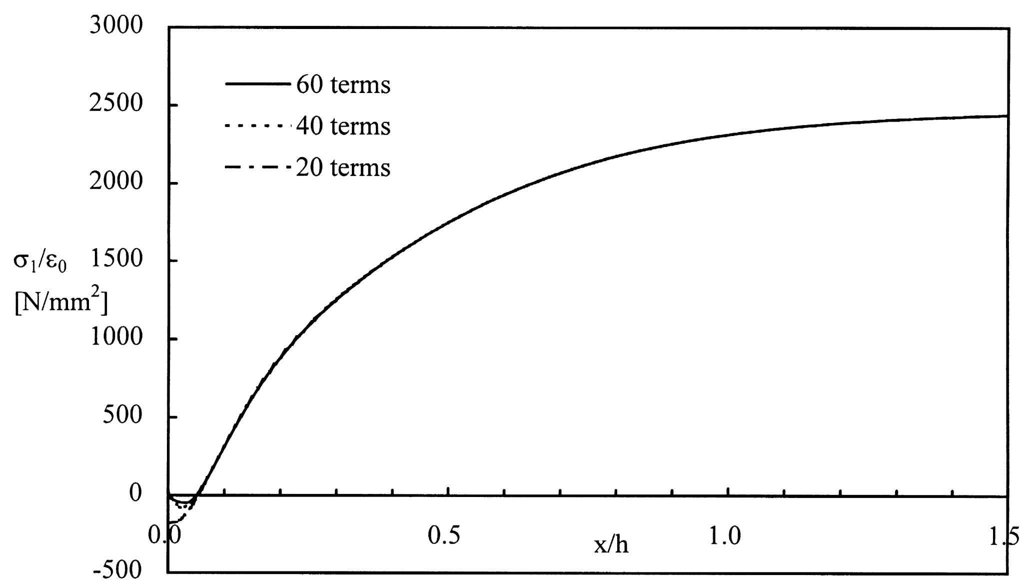

Fig. 5. In-plane stress σ_1/ϵ_0 at the middle surface of the 0-layer.

boundary layer zone in this case is localized to the region of a distance 1.5 h from the edge. Figs. 4 and 5 show the in-plane stress distribution σ_1/ϵ_0 at the middle surface of the laminate and at the middle surface of the 0-layer, respectively. The stress σ_1/ϵ_0 approaches to a constant value at about $x_1 > 1.5\ h$. The traction-free edge condition requires σ_1/ϵ_0 be zero at $x_1 = 0$. The condition is satisfied as the expansion terms increase. The stress near the free edge shows numerical oscillations which are common to solutions obtained via eigenfunction expansions involving complex eigenvalues. In Figs. 2 and 3 the computed values of interlaminar stresses, as $x_1 \to 0$, grow with the terms taken in the expansion. The free-edge normal and shear stresses at the 0/90 interfaces are not vanishing due to stress singularities. It should be noted that the exact nature of the singularity cannot be determined numerically in the present context since the singular behavior at the free edge interfaces was not considered in the analysis. Applications of the state space approach and transfer matrix in studying the stress singularities in multilayered laminates requires a continuing study.

Acknowledgements

We thank the support from the National Science Council of Republic of China through grants NSC87-2211-E006-051 and NSC87-2211-E006-065.

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